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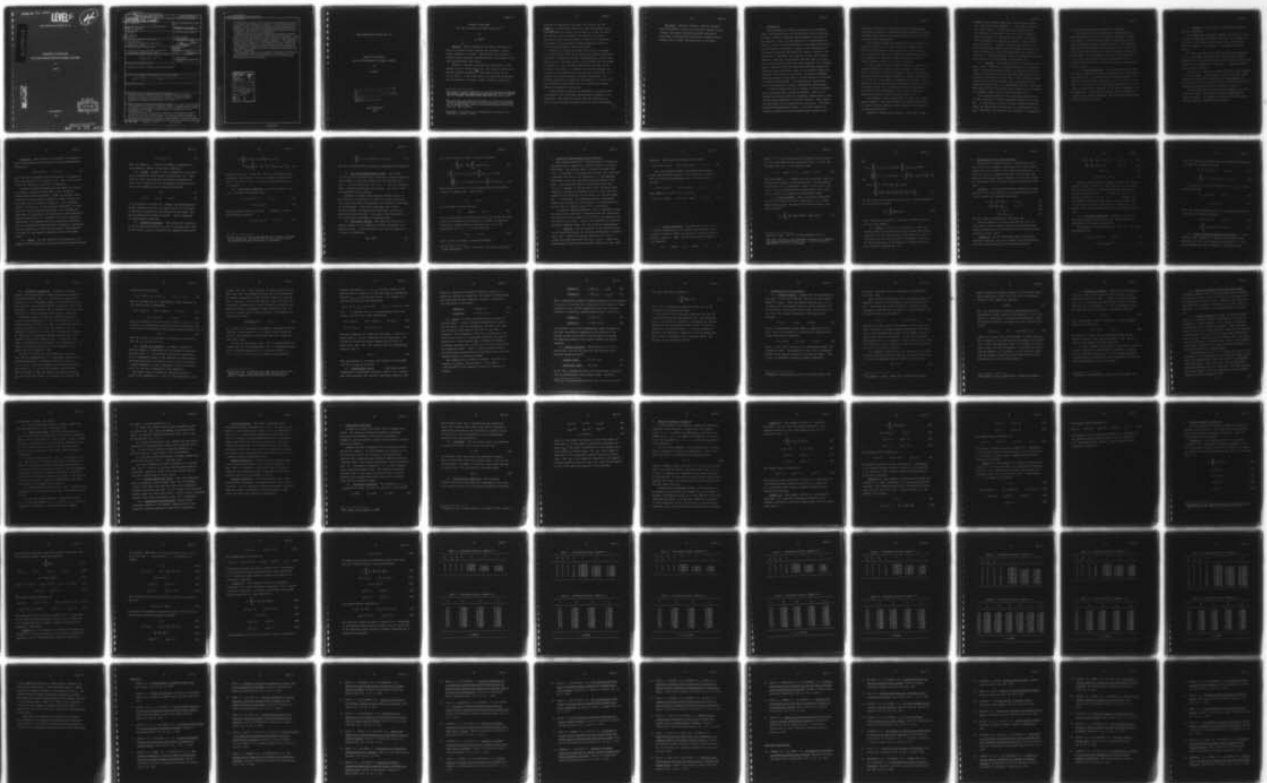
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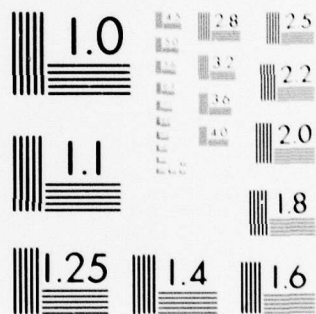
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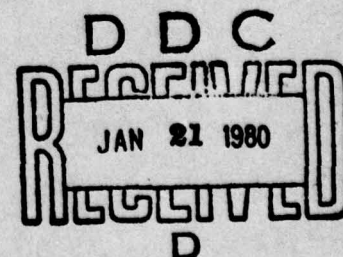
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of integration, the state, the control, and the parameter are required to satisfy  $n$  scalar differential equations. Problem P2 differs from Problem P1 in that the state, the control, and the parameter are required to satisfy  $k$  additional scalar relations along the interval of integration. Algorithms of the sequential gradient-restoration type are given for both Problem P1 and Problem P2.

Problem P2 enlarges the number and variety of problems of optimal control which can be treated by gradient-restoration algorithms. Indeed, by suitable transformations, almost every known problem of optimal control can be brought into the scheme of Problem 2. This statement applies, for instance, to the following situations: (i) problems with control equality constraints, (ii) problems with state equality constraints, (iii) problems with state-derivative equality constraints, (iv) problems with control inequality constraints, (v) problems with state inequality constraints, and (vi) problems with state-derivative inequality constraints.

Eight numerical examples are presented to illustrate the performance of the algorithms associated with Problem P1 and Problem P2. The numerical results show the feasibility as well as the convergence characteristics of these algorithms.

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GRADIENT ALGORITHMS  
FOR THE OPTIMIZATION OF DYNAMIC SYSTEMS

by

A. MIELE

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Gradient Algorithms  
for the Optimization of Dynamic Systems<sup>1,2</sup>

by

A. MIELE<sup>3</sup>



Abstract. Recent advances in the area of gradient methods for optimal control problems are reviewed. Single-subarc problems are treated. Specifically, two classes of optimal control problems, called Problem P1 and Problem P2 for easy identification, are solved.

Problem P1 consists of minimizing a functional  $I$  which depends on the  $n$ -vector state  $x(t)$ , the  $m$ -vector control  $u(t)$ , and the  $p$ -vector parameter  $\pi$ <sup>3.14</sup>. The state is given at the initial point. At the final point, the state and the parameter are required to satisfy  $q$  scalar relations. Along the

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<sup>1</sup>This work is being contributed to the book Advances in Control and Dynamic Systems: Theory and Applications, Vol. 16, Edited by C.T. Leondes, Academic Press, New York, New York, 1979.

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<sup>3</sup>Professor of Astronautics and Mathematical Sciences, Rice University, Houston, Texas.



interval of integration, the state, the control, and the parameter are required to satisfy  $n$  scalar differential equations. <sup>cont.</sup> Problem P2 differs from Problem P1 in that the state, the control, and the parameter are required to satisfy  $k$  additional scalar relation along the interval of integration. Algorithms of the sequential gradient-restoration type are given for both Problem P1 and Problem P2.

Problem P2 enlarges the number and variety of problems of optimal control which can be treated by gradient-restoration algorithms. Indeed, by suitable transformations, almost every known problem of optimal control can be brought into the scheme of Problem P2. This statement applies, for instance, to the following situations: (i) problems with control equality constraints, (ii) problems with state equality constraints, (iii) problems with state-derivative equality constraints, (iv) problems with control inequality constraints, (v) problems with state inequality constraints, and (vi) problems with state-derivative inequality constraints.

Eight numerical examples are presented to illustrate the performance of the algorithms associated with Problem P1 and Problem P2. The numerical results show the feasibility as well as the convergence characteristics of these algorithms.





Key Words. Numerical analysis, numerical methods, computing methods, calculus of variations, optimal control, gradient algorithms, sequential gradient-restoration algorithms, nondifferential constraints, bounded control, bounded state, bounded time-derivative of the state.

## 1. Introduction

In every branch of science, engineering, and economics, there exist systems which are controllable, that is, they can be made to behave in different ways depending on the will of the operator. Every time the operator of a system exerts an option, a choice in the distribution of the quantities controlling the system, he produces a change in the distribution of the states occupied by the system and, hence, a change in the final state. Therefore, it is natural to pose the following question: Among all the admissible options, what is the particular option which renders the system optimum? As an example, what is the option which minimizes the difference between the final value and the initial value of an arbitrarily specified function of the state of the system? The body of knowledge covering problems of this type is called calculus of variations or optimal control theory. As stated before, applications occur in every field of science, engineering, and economics.

It must be noted that only a minority of current problems can be solved by purely analytical methods. Hence, it is important to develop numerical techniques enabling one to solve optimal control problems on a digital computer. These numerical techniques can be classified into two groups: first-order methods and second-order methods. First-order methods (or gradient methods) are those techniques which employ at most

the first derivatives of the functions under consideration. Second-order methods (or quasilinearization methods) are those techniques which employ at most the second derivatives of the functions under consideration.

Both gradient methods and quasilinearization methods require the solution of a linear, two-point or multi-point boundary-value problem at every iteration. This being the case, progress in the area of numerical methods for differential equations is essential to the efficient solution of optimal control problems on a digital computer.

In this paper, we review recent advances in the area of gradient methods for optimal control problems. Because of space limitations, we make no attempt to cover every possible technique and every possible approach, a material impossibility in view of the large number of publications available. Thus, except for noting the early work performed by Kelley (Refs. 1-2) and Bryson (Refs. 3-6), we devote the body of the paper to a review of the work performed in recent years by the Aero-Astronautics Group of Rice University (Refs. 7-34).

Also because of space limitations, we treat only single-subarc problems. More specifically, we consider two classes of optimal control problems, called Problem P1 and Problem P2 for easy identification.

Problem P1 consists of minimizing a functional  $I$  which



depends on the  $n$ -vector state  $x(t)$ , the  $m$ -vector control  $u(t)$ , and the  $p$ -vector parameter  $\pi$ . The state is given at the initial point. At the final point, the state and the parameter are required to satisfy  $q$  scalar relations. Along the interval of integration, the state, the control, and the parameter are required to satisfy  $n$  scalar differential equations. Problem P2 differs from Problem P1 in that the state, the control, and the parameter are required to satisfy  $k$  additional scalar relations along the interval of integration. Algorithms of the sequential gradient-restoration type are given for both Problem P1 and Problem P2.

1.1. Approach. The approach taken is a sequence of two-phase cycles, composed of a gradient phase and a restoration phase. The gradient phase involves one iteration and is designed to decrease the value of the functional, while the constraints are satisfied to first order. The restoration phase involves one or more iterations, and is designed to force constraint satisfaction to a predetermined accuracy, while the norm squared of the variations of the control and the parameter is minimized, subject to the linearized constraints.

The principal property of the algorithms presented here is that a sequence of feasible suboptimal solutions is produced. In other words, at the end of each gradient-restoration cycle, the constraints are satisfied to a predetermined accuracy. Therefore, the values of the functional  $I$  corresponding



to any two elements of the sequence are comparable.

The stepsize of the gradient phase is determined by a one-dimensional search on the augmented functional  $J$ , while the stepsize of the restoration phase is obtained by a one-dimensional search on the constraint error  $P$ . The gradient stepsize and the restoration stepsize are chosen so that the restoration phase preserves the descent property of the gradient phase. As a consequence, the value of the functional  $I$  at the end of any complete gradient-restoration cycle is smaller than the value of the same functional at the beginning of that cycle.

1.2. Time Normalization. A time normalization is used in order to simplify the numerical computations. Specifically, the actual time  $\theta$  is replaced by the normalized time  $t = \theta/\tau$ , which is defined in such a way that  $t = 0$  at the initial point and  $t = 1$  at the final point. The actual final time  $\tau$ , if it is free, is regarded as a component of the vector parameter  $\pi$  to be optimized. In this way, an optimal control problem with variable final time is converted into an optimal control problem with fixed final time.

1.3. Notation. In this paper, vector-matrix notation is used for conciseness.

Let  $t$  denote the independent variable, and let  $x(t)$ ,  $u(t)$ ,  $\pi$  denote the dependent variables. The time  $t$  is a scalar, the state  $x(t)$  is an  $n$ -vector, the control  $u(t)$  is an  $m$ -vector, and the parameter  $\pi$  is a  $p$ -vector. All vectors are column vectors.

Let  $h(x, u, \pi, t)$  denote a scalar function of the arguments  $x, u, \pi, t$ . The symbol  $h_x$  denotes the  $n$ -vector function whose components are the partial derivatives of the scalar function  $h$  with respect to the components of the vector  $x$ . Analogous definitions hold for  $h_u$  and  $h_\pi$ .

Let  $\omega(x, u, \pi, t)$  denote an  $r$ -vector function of the arguments  $x, u, \pi, t$ . The symbol  $\omega_x$  denotes the  $n \times r$  matrix function whose elements are the partial derivatives of the components of the vector function  $\omega$  with respect to the components of the vector  $x$ . Analogous definitions hold for the symbols  $\omega_u$  and  $\omega_\pi$ .

The dot sign denotes derivative with respect to the time, that is,  $\dot{x} = dx/dt$ . The symbol  $T$  denotes transposition of vector or matrix. The subscript 0 denotes the initial point, and the subscript 1 denotes the final point.

1.4. Outline. Section 2 contains the statements of Problem P1 and Problem P2. Section 3 gives a description of the sequential gradient-restoration algorithm. Section 4 discusses the determinations of the basic functions for the gradient phase and the restoration phase. Section 5 considers the determination of the stepsizes for the gradient phase and the restoration phase. A summary of the sequential gradient-restoration algorithm is presented in Section 6. The experimental conditions are given in Section 7. The numerical examples for Problem P1 are given in Section 8; and the numerical examples for Problem P2 are given in Section 9. Finally, the discussion and the conclusions are presented in Section 10.

## 2. Statement of the Problems

Problem Pl. This problem consists of minimizing the functional

$$I = \int_0^1 f(x, u, \pi, t) dt + [g(x, \pi, t)]_1, \quad (1)$$

with respect to the state  $x(t)$ , the control  $u(t)$ , and the parameter  $\pi$  which satisfy the differential constraints

$$\dot{x} - \phi(x, u, \pi, t) = 0, \quad 0 \leq t \leq 1, \quad (2)$$

the initial conditions

$$x(0) = \text{given}, \quad (3)$$

and the final conditions

$$[\psi(x, \pi, t)]_1 = 0. \quad (4)$$

In Eqs. (1)-(4), the quantities  $I, f, g$  are scalar, the function  $\phi$  is an  $n$ -vector, and the function  $\psi$  is a  $q$ -vector. Eqs.

(2)-(4) constitute the feasibility equations for Problem Pl.



Problem P2. This problem is an extension of Problem P1, which arises because of the inclusion of the nondifferential constraints

$$S(x, u, \pi, t) = 0, \quad 0 \leq t \leq 1, \quad (5)$$

to be satisfied everywhere along the interval of integration. Here, the function  $S$  is a  $k$ -vector,  $k \leq m$ . Eqs. (2)-(5) constitute the feasibility equations of Problem P2.

Problem P2 enlarges dramatically the number and variety of problems of optimal control which can be treated by gradient-restoration algorithms. Indeed, by suitable transformations, almost every known problem of optimal control can be brought into the scheme of Problem P2. This statement applies, for instance, to the following situations: (i) problems with control equality constraints, (ii) problems with state equality constraints, (iii) problems with state-derivative equality constraints, (iv) problems with control inequality constraints, (v) problems with state inequality constraints, and (vi) problems with state-derivative inequality constraints. For an illustration of the scope and range of applicability of Problem P2, the reader is referred to Ref. 19 and Refs. 25-29.

2.1. Remark. For both Problem P1 and Problem P2, the number of final conditions  $q$  must satisfy the following relation:

$$q \leq n + p_* \leq n + p, \quad (6)$$

where the symbol  $p_*$  denotes the number of components of the parameter  $\pi$  present in the final conditions.

2.2. Remark. Problem P1 can be regarded as a particular case of Problem P2, which arises by deleting Eq. (5). This being the case, the analytical derivations presented here refer only to Problem P2. The corresponding analytical derivations for Problem P1 can be obtained by setting

$$S \equiv 0, \quad (7)$$

$$S_x \equiv 0, \quad S_u \equiv 0, \quad S_\pi \equiv 0 \quad (8)$$

in the equations of Problem P2. However, the differentiation between Problems P1 and P2 is invoked later on in the paper, in the section dealing with the solution of the linear, two-point boundary-value problem (LTP-BVP). This is necessary for computational efficiency.

2.3. Augmented Functional. From calculus of variations, it can be seen that Problem P2 is one of the Bolza type, which can be recast as that of minimizing the augmented functional<sup>4</sup>

$$\begin{aligned}
 J &= \int_0^1 [f + \lambda^T(\dot{x} - \phi) + \rho^T S] dt + (g + \mu^T \psi)_1 \\
 &= (\lambda^T x)_0^1 + \int_0^1 (f - \lambda^T \phi + \rho^T S - \dot{\lambda}^T x) dt + (g + \mu^T \psi)_1, \quad (9)
 \end{aligned}$$

subject to (2)-(5). In Eq. (9),  $\lambda(t)$  is a variable Lagrange multiplier (an  $n$ -vector),  $\rho(t)$  is a variable Lagrange multiplier (a  $k$ -vector), and  $\mu$  is a constant Lagrange multiplier (a  $q$ -vector).

2.4. First-Order Conditions. Let the multipliers  $\lambda(t)$ ,  $\rho(t)$ ,  $\mu$  be chosen consistently with

$$\dot{\lambda} - f_x + \phi_x \lambda - S_x \rho = 0, \quad 0 \leq t \leq 1, \quad (10)$$

$$(\lambda + g_x + \psi_x \mu)_1 = 0. \quad (11)$$

Then, the optimal control  $u(t)$  and parameter  $\pi$  satisfy the following relations:

$$f_u - \phi_u \lambda + S_u \rho = 0, \quad 0 \leq t \leq 1, \quad (12)$$

---

<sup>4</sup>In Eq. (9), it is tacitly assumed that the initial conditions (3) are satisfied. The second form of Eq. (9) arises after the customary integration by parts is performed.

$$\int_0^1 (f_{\pi} - \phi_{\pi} \lambda + S_{\pi} \rho) dt + (g_{\pi} + \psi_{\pi} \mu)_1 = 0. \quad (13)$$

Eqs. (10)-(13) constitute the optimality conditions for Problem P2.

2.5. Two-Point Boundary-Value Problem. The system (2)-(5) and (10)-(13) constitutes a nonlinear, two-point boundary-value problem in which the unknowns are the functions  $x(t)$ ,  $u(t)$ ,  $\pi$  and the multipliers  $\lambda(t)$ ,  $\rho(t)$ ,  $\mu$ . Only for particular cases, closed-form solutions are possible. In general, numerical methods must be employed.

Depending on whether these numerical methods employ at most the first derivatives or at most the second derivatives of the functions under consideration, two classes of algorithms can be developed: first-order algorithms (also called gradient methods) and second-order algorithms (also called quasilinearization methods). As stated in the introduction, only first-order algorithms are considered here.

2.6. Performance Indexes. When solving Problem P2 on a digital computer, it is necessary to define convergence in a numerical sense. In this connection, let the norm squared of a vector  $y$  be defined as

$$N(y) = y^T y. \quad (14)$$



Let  $P$  and  $Q$  denote the scalar performance indexes<sup>5</sup>

$$P = \int_0^1 N(\dot{x} - \phi) dt + \int_0^1 N(S) dt + N(\psi)_1, \quad (15)$$

$$Q = \int_0^1 N(\dot{\lambda} - f_x + \phi_x \lambda - S_x \rho) dt + \int_0^1 N(f_u - \phi_u \lambda + S_u \rho) dt \\ + N \left[ \int_0^1 (f_\pi - \phi_\pi \lambda + S_\pi \rho) dt + (g_\pi + \psi_\pi \mu)_1 \right] + N(\lambda + g_x + \psi_x \mu)_1, \quad (16)$$

which measure the errors in the constraints and the optimality conditions, respectively. Observe that

$$P = 0, \quad Q = 0, \quad (17)$$

for the optimal solution and that

$$P > 0 \quad \text{and/or} \quad Q > 0, \quad (18)$$

for any approximation to the optimal solution. This being the case, numerical convergence can be defined as follows: an iterative algorithm is stopped whenever functions  $x(t)$ ,  $u(t)$ ,  $\pi$  and multipliers  $\lambda(t)$ ,  $\rho(t)$ ,  $\mu$  are found such that

$$P \leq \varepsilon_1, \quad Q \leq \varepsilon_2, \quad (19)$$

where  $\varepsilon_1$  and  $\varepsilon_2$  are small, preselected numbers.

<sup>5</sup>In Eq. (15), it is tacitly assumed that the initial conditions (3) are satisfied.

### 3. Sequential Gradient-Restoration Algorithm

The technique employed is characterized by a sequence of two-phase cycles, composed of a gradient phase and a restoration phase. The gradient phase is started only when Ineq. (19-1) is satisfied; it involves one iteration and is designed to decrease the value of the functional  $I$  or the augmented functional  $J$ , while the constraints are satisfied to first order. The restoration phase is started only when Ineq. (19-1) is violated; it involves one or more iterations, each designed to decrease the constraint error  $P$ , while the norm squared of the variations of the control  $u(t)$  and the parameter  $\pi$  is minimized. The restoration phase is terminated whenever the constraints are satisfied to a predetermined accuracy, that is, whenever Ineq. (19-1) is satisfied.

A complete gradient-restoration cycle is designed so that the value of the functional  $I$  decreases while the constraints are satisfied to the accuracy (19-1) both at the beginning and at the end of the cycle. Finally, the algorithm as a whole is terminated whenever Ineqs. (19) are satisfied simultaneously.

3.1. Notation. For any iteration of the gradient phase or the restoration phase, the following terminology is adopted:  $x(t)$ ,  $u(t)$ ,  $\pi$  denote the nominal functions;  $\tilde{x}(t)$ ,  $\tilde{u}(t)$ ,  $\tilde{\pi}$  denote the varied functions; and  $\Delta x(t)$ ,  $\Delta u(t)$ ,  $\Delta \pi$  denote the displacements leading from the nominal functions to the varied

functions. These quantities satisfy the relations

$$\tilde{x}(t) = x(t) + \Delta x(t), \quad \tilde{u}(t) = u(t) + \Delta u(t), \quad \tilde{\pi} = \pi + \Delta \pi. \quad (20)$$

Let  $\alpha$  be a positive number representing the stepsize (either the gradient stepsize or the restoration stepsize). Then, we define the displacements per unit of stepsize as follows:

$$A(t) = \Delta x(t)/\alpha, \quad B(t) = \Delta u(t)/\alpha, \quad C = \Delta \pi/\alpha. \quad (21)$$

Upon combining (20) and (21), we see that

$$\tilde{x}(t) = x(t) + \alpha A(t), \quad \tilde{u}(t) = u(t) + \alpha B(t), \quad \tilde{\pi} = \pi + \alpha C. \quad (22)$$

3.2. Desired Properties. The functions  $\Delta x(t)$ ,  $\Delta u(t)$ ,  $\Delta \pi$  must be determined so as to produce some desirable effect at every iteration, namely, the decrease of the functionals  $I$ , and/or  $J$ , and/or  $P$ . Thus, the following descent properties are required:

$$\tilde{I} < I, \quad \text{and/or} \quad \tilde{J} < J, \quad \text{and/or} \quad \tilde{P} < P, \quad (23)$$

where  $I, J, P$  are associated with the nominal functions and  $\tilde{I}, \tilde{J}, \tilde{P}$  are associated with the varied functions. In turn, the functions  $A(t), B(t), C$  are chosen so that

$$\delta I < 0, \quad \text{and/or} \quad \delta J < 0, \quad \text{and/or} \quad \delta P < 0, \quad (24)$$

where the symbol  $\delta(\dots)$  denotes the first variation. Then, by choosing the stepsize  $\alpha$  sufficiently small, the satisfaction of relations (23) is guaranteed. Ineqs. (23-1), (23-2) and (24-1), (24-2) characterize the gradient phase, while Ineqs. (23-3) and (24-3) characterize the restoration phase.

3.3. First Variations. Next, we give the expressions for the first variations of the functionals  $I, J, P$ ; after simple manipulations, omitted for the sake of brevity, they take the form<sup>6,7</sup>

$$\delta I/\alpha = \int_0^1 (f_x^T A + f_u^T B + f_\pi^T C) dt + (g_x^T A + g_\pi^T C)_1, \quad (25)$$

---

<sup>6</sup> Implicit in Eqs. (25)-(27) is the assumption  $A(0) = 0$ .

<sup>7</sup> The first variation of the augmented functional  $J$  is computed by varying the functions  $x(t), u(t), \pi$ , while holding the multipliers  $\lambda(t), \rho(t), \mu$  unchanged.



and

$$\delta J/\alpha = \int_0^1 (-\dot{\lambda} + f_x - \phi_x \lambda + S_x \rho)^T A dt + \int_0^1 (f_u - \phi_u \lambda + S_u \rho)^T B dt \\ + \left[ \int_0^1 (f_\pi - \phi_\pi \lambda + S_\pi \rho) dt + (g_\pi + \psi_\pi \mu)_1 \right]^T C + \left[ (\lambda + g_x + \psi_x \mu)^T A \right]_1, \quad (26)$$

and

$$\delta P/2\alpha = \int_0^1 (\dot{x} - \phi)^T (\dot{A} - \phi_x^T A - \phi_u^T B - \phi_\pi^T C) dt \\ + \int_0^1 S^T (S_x^T A + S_u^T B + S_\pi^T C) dt + \left[ \psi^T (\psi_x^T A + \psi_\pi^T C) \right]_1. \quad (27)$$

For the purposes of this paper, Eqs. (25)-(27) must be completed by the following relation:

$$K/\alpha^2 = \int_0^1 B^T B dt + C^T C, \quad (28)$$

which constitutes a measure of the overall change of the control and the parameter.

3.4. Remark. Clearly, every iteration of either the gradient phase or the restoration phase includes two distinct operations: (a) the determination of functions  $A(t)$ ,  $B(t)$ ,  $C$  consistent with the first variation requirements (24); and (b) the determination of the stepsize  $\alpha$  consistent with the total variation requirements (23).

#### 4. Determination of the Basic Functions

There exist an infinite number of combinations of functions  $A(t)$ ,  $B(t)$ ,  $C$  capable of satisfying the first-variation inequalities (24), subject to the linearized constraints. In order to arrive at a unique combination of functions, some additional requirement must be imposed. This is done through the formulation of the following auxiliary minimization problems.

Problem P3. For the gradient phase, minimize the linear functional (25), with respect to the perturbations  $A(t)$ ,  $B(t)$ ,  $C$  which satisfy the linearized constraints

$$\dot{A} - \phi_x^T A - \phi_u^T B - \phi_\pi^T C = 0, \quad 0 \leq t \leq 1, \quad (29)$$

$$S_x^T A + S_u^T B + S_\pi^T C = 0, \quad 0 \leq t \leq 1, \quad (30)$$

$$A(0) = 0, \quad (31)$$

$$(\psi_x^T A + \psi_\pi^T C)_1 = 0, \quad (32)$$

and the quadratic isoperimetric constraint (28).

Note the absence of forcing terms from Eqs. (29)-(32). This implies that the nominal functions characterizing the gradient phase satisfy the constraints (2)-(5) within the pre-selected accuracy (19-1).

Problem P4. For the restoration phase, minimize the quadratic functional (28), with respect to the perturbations  $A(t)$ ,  $B(t)$ ,  $C$  which satisfy the linearized constraints

$$\dot{A} - \phi_x^T A - \phi_u^T B - \phi_\pi^T C + (\dot{x} - \phi) = 0, \quad 0 \leq t \leq 1, \quad (33)$$

$$S_x^T A + S_u^T B + S_\pi^T C + S = 0, \quad 0 \leq t \leq 1, \quad (34)$$

$$A(0) = 0, \quad (35)$$

$$(\psi_x^T A + \psi_\pi^T C + \psi)_1 = 0. \quad (36)$$

Note that forcing terms are absent from Eq. (35), but are present in Eqs. (33), (34), (36). This implies that the nominal functions characterizing the restoration phase satisfy the initial conditions (3), but violate one or more of the remaining constraints (2), (4), (5). Indeed, it is the purpose of the restoration phase to correct these violations, while causing the least possible disturbance in the system. This is the significance of the least-square criterion (28).

4.1. First-Order Conditions. Problems P3 and P4 are variational problems of the Bolza type, each governed by a different set of optimality conditions.

For Problem P3, let the multipliers  $\lambda(t)$ ,  $\rho(t)$ ,  $\mu$  be chosen consistently with

$$\dot{\lambda} - f_x + \phi_x \lambda - S_x \rho = 0, \quad 0 \leq t \leq 1, \quad (37)$$

$$(\lambda + g_x + \psi_x \mu)_1 = 0. \quad (38)$$

Then, the control perturbation  $B(t)$  and parameter perturbation  $C$  satisfy the following relations:

$$B + f_u - \phi_u \lambda + S_u \rho = 0, \quad 0 \leq t \leq 1, \quad (39)$$

$$C + \int_0^1 (f_\pi - \phi_\pi \lambda + S_\pi \rho) dt + (g_\pi + \psi_\pi \mu)_1 = 0. \quad (40)$$

For Problem P4, let the multipliers  $\lambda(t)$ ,  $\rho(t)$ ,  $\mu$  be chosen consistently with

$$\dot{\lambda} + \phi_x \lambda - S_x \rho = 0, \quad 0 \leq t \leq 1, \quad (41)$$

$$(\lambda + \psi_x \mu)_1 = 0. \quad (42)$$

Then, the control perturbation  $B(t)$  and parameter perturbation  $C$  satisfy the following relations:

$$B - \phi_u \lambda + S_u \rho = 0, \quad 0 \leq t \leq 1, \quad (43)$$

$$C + \int_0^1 (-\phi_\pi \lambda + S_\pi \rho) dt + (\psi_\pi \mu)_1 = 0. \quad (44)$$

4.2. Linear, Two-Point Boundary-Value Problem. For the gradient phase, Problem P3 is governed by the feasibility equations (29)-(32) and the optimality conditions (37)-(40). For the restoration phase, Problem P4 is governed by the



feasibility equations (33)-(36) and the optimality conditions (41)-(44). The form of these feasibility equations and optimality conditions is such that the systems governing Problems P3 and P4 can be embedded in a single linear system. For compactness, as well as to facilitate programming, this point of view is taken here, and the single system governing both the gradient phase and the restoration phase is written as follows:

$$\dot{A} - \phi_x^T A - \phi_u^T B - \phi_\pi^T C + k_r (\dot{x} - \phi) = 0, \quad 0 \leq t \leq 1, \quad (45)$$

$$S_x^T A + S_u^T B + S_\pi^T C + k_r S = 0, \quad 0 \leq t \leq 1, \quad (46)$$

$$A(0) = 0, \quad (47)$$

$$(\psi_x^T A + \psi_\pi^T C + k_r \psi)_1 = 0, \quad (48)$$

and

$$\dot{\lambda} - k_g f_x + \phi_x \lambda - S_x \rho = 0, \quad 0 \leq t \leq 1, \quad (49)$$

$$(\lambda + k_g g_x + \psi_x \mu)_1 = 0, \quad (50)$$

$$B + k_g f_u - \phi_u \lambda + S_u \rho = 0, \quad 0 \leq t \leq 1, \quad (51)$$

$$C + \int_0^1 (k_g f_\pi - \phi_\pi \lambda + S_\pi \rho) dt + (k_g g_\pi + \psi_\pi \mu)_1 = 0. \quad (52)$$

The constants  $k_g$  and  $k_r$  appearing in (45)-(52) take the

following values:

$$\text{gradient phase,} \quad k_g = 1, \quad k_r = 0; \quad (53)$$

$$\text{restoration phase,} \quad k_g = 0, \quad k_r = 1. \quad (54)$$

For given nominal functions  $x(t)$ ,  $u(t)$ ,  $\pi$  and given constants  $k_g$  and  $k_r$ , Eqs. (45)-(52) define the functions  $A(t)$ ,  $B(t)$ ,  $C$  and the multipliers  $\lambda(t)$ ,  $\rho(t)$ ,  $\mu$ . As can be seen, we are in the presence of a linear, two-point boundary-value problem (LTP-BVP), which can be solved independently of the value assigned to the stepsize  $\alpha$ .

In principle, the LTP-BVP (45)-(52) can be discussed simultaneously for both Problem P1 and Problem P2. However, for computational efficiency, it is better to separate the discussion of Problem P1 from that of Problem P2. This is because the LTP-BVP for Problem P1 can be solved executing  $q+1$  independent sweeps of the differential system, while the LTP-BVP for Problem P2 requires the execution of  $n+p+1$  independent sweeps. Here,  $q$  is the number of final conditions,  $n$  is the dimension of the state vector, and  $p$  is the dimension of the parameter vector.

4.3. LTP-BVP for Problem P1. We employ a backward-forward integration scheme in combination with the method of particular solutions (Ref. 7). The technique requires the execution of  $q+1$  independent sweeps of the differential system (45)-(52), each characterized by a different value of the multiplier  $\mu$ . Note that Eq. (46) is deleted and that the simplifications (7)-(8) are invoked in the remaining equations.

The generic sweep is started by assigning particular values to the components of  $\mu$ ; then, the multiplier  $\lambda(1)$  is obtained from Eq.(50). Next, Eq. (49) is integrated backward to obtain the function  $\lambda(t)$ . With  $\lambda(t)$  known, Eq. (51) is employed to obtain  $B(t)$ , and Eq. (52) is employed to compute  $C$ . Then,  $A(t)$  is obtained by forward integration of (45), subject to the initial condition (47). In this way, the sweep is completed: for the arbitrary value assigned to  $\mu$ , it leads to the satisfaction of all of the equations of the system (45)-(52), except Eq. (48).

In order to satisfy Eq. (48) and because the system (45)-(52) is nonhomogeneous,  $q+1$  independent sweeps must be executed employing  $q+1$  different multiplier vectors  $\mu_i$ ,  $i=1, \dots, q+1$ . The first  $q$  sweeps are performed by choosing the vectors  $\mu_1, \dots, \mu_q$  to be the columns of the identity matrix of order  $q$ . The last sweep is executed by choosing  $\mu_{q+1}$  to be the null vector. As a result, one generates the

functions and multipliers

$$A_i(t), B_i(t), C_i, \lambda_i(t), \mu_i, \quad i = 1, \dots, q+1. \quad (55)$$

Now, we introduce the  $q+1$  undetermined, scalar constants  $k_i$  and form the linear combinations

$$A(t) = \sum k_i A_i(t), \quad B(t) = \sum k_i B_i(t), \quad C = \sum k_i C_i, \quad (56)$$

$$\lambda(t) = \sum k_i \lambda_i(t), \quad \mu = \sum k_i \mu_i, \quad (57)$$

where the summations are taken over the index  $i$ . The  $q+1$  coefficients  $k_i$  are obtained by forcing the linear combinations (56) to satisfy Eq. (48), together with the normalization condition (Ref. 7)

$$\sum k_i = 1. \quad (58)$$

Once the constants  $k_i$  are known, the solution of the LTP-BVP (45)-(52) is given by (56)-(57).

4.4. LTP-BVP for Problem P2. We employ a forward integration scheme in combination with the method of particular solutions (Ref. 7). The technique requires the execution of  $n+p+1$  independent sweeps of the differential system (45)-(52), each characterized by a different value of the  $(n+p)$ -vector  $\sigma$ , whose components are the  $n$  components of the initial multiplier  $\lambda(0)$  and the  $p$  components of the parameter  $C$ .

The generic sweep is started by assigning particular values to the components of  $\sigma$ , that is, the components of the



vectors  $\lambda(0)$  and  $C$ . Note that  $A(0)$  is known, because of (47). Then,  $A(t)$  and  $\lambda(t)$ , together with  $B(t)$  and  $\rho(t)$ , are obtained by forward integration of (45) and (49), subject to (46) and (51). Note that, at each time station  $t$ , Eqs. (46) and (51) constitute a system of  $m+k$  linear relations in which the unknowns are the  $m+k$  components of the vectors  $B(t)$  and  $\rho(t)$ . For this system to have a unique solution, the following disequation must hold:<sup>8</sup>

$$\det \begin{bmatrix} S_u^T & S_u \end{bmatrix} \neq 0, \quad 0 \leq t \leq 1. \quad (59)$$

As a result of the procedure, the sweep is completed: for the arbitrary value assigned to  $\sigma$ , it leads to the satisfaction of all of the equations of the system (45)-(52), except Eqs. (48), (50), (52).

In order to satisfy Eqs. (48), (50), (52) and because the system (45)-(52) is nonhomogeneous,  $n+p+1$  independent sweeps must be executed employing  $n+p+1$  different vectors  $\sigma_i$ ,  $i=1, \dots, n+p+1$ . The first  $n+p$  sweeps are performed by

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<sup>8</sup>Disequation (59) is obtained from (46) and (51) after elimination of  $B(t)$ . The resulting linear equation in  $\rho(t)$  admits a unique solution providing (59) is satisfied.

choosing the vectors  $\sigma_1, \dots, \sigma_{n+p}$  to be the columns of the identity matrix of order  $n+p$ . The last sweep is executed by choosing  $\sigma_{n+p+1}$  to be the null vector. As a result, one generates the functions and multipliers

$$A_i(t), B_i(t), C_i, \lambda_i(t), \rho_i(t), \quad i=1, \dots, n+p+1. \quad (60)$$

Now, we introduce the  $n+p+1$  undetermined, scalar constants  $k_i$  and form the linear combinations

$$A(t) = \sum k_i A_i(t), \quad B(t) = \sum k_i B_i(t), \quad C = \sum k_i C_i, \quad (61)$$

$$\lambda(t) = \sum k_i \lambda_i(t), \quad \rho(t) = \sum k_i \rho_i(t), \quad (62)$$

where the summations are taken over the index  $i$ . The  $n+p+1$  coefficients  $k_i$  and the  $q$  components of the multiplier  $\mu$  are obtained by forcing the linear combinations (61)-(62) to satisfy Eqs. (48), (50), (52) together with the normalization condition (Ref. 7)

$$\sum k_i = 1. \quad (63)$$

Once the constants  $k_i$  are known, the solution of the LTP-BVP (45)-(52) is given by (61)-(62).

4.5. Computational Effort. Each sweep involves integrating  $2n$  differential equations, that is, the  $n$  linearized state equations (45) and the  $n$  multiplier equations (49).

Since  $q+1$  sweeps are involved in Problem P1 and  $n+p+1$  sweeps are involved in Problem P2, the amount of computational work performed per each iteration, gradient or restorative, is proportional to the factor:

$$\text{Problem P1,} \quad w = 2n(q+1); \quad (64)$$

$$\text{Problem P2,} \quad w = 2n(n+p+1). \quad (65)$$

4.6. Remark. For both the gradient phase and the restoration phase, a linear, two-point boundary-value problem must be solved. Once the constants  $k_i$  are known, the composite solution is obtained via (56)-(57) or (61)-(62). A drawback of this procedure is that the  $q+1$  particular solutions (55) or the  $n+p+1$  particular solutions (60) must be stored at  $N+1$  time stations (here,  $N$  denotes the number of integration subintervals, so that  $\Delta t = 1/N$  is the magnitude of the integration step). Hence, a storage problem arises if the system under consideration is relatively large, while the computer memory is relatively limited.

This drawback can be offset as follows. Once the constants  $k_i$  are known, the multiplier  $\mu$  or the vector  $\sigma = [\lambda^T(0), C^T]^T$  of the composite solution is computed as follows:

$$\text{Problem P1,} \quad \mu = [k_1, k_2, \dots, k_q]^T; \quad (66)$$

$$\text{Problem P2,} \quad \sigma = [k_1, k_2, \dots, k_{n+p}]^T. \quad (67)$$

Then, a supplementary sweep is executed according to the procedure of Section 4.3 or Section 4.4. Clearly, the total number of sweeps increases by one, and the computational work per iteration, gradient or restorative, becomes proportional to the factor:

$$\text{Problem P1,} \quad w = 2n(q + 2); \quad (68)$$

$$\text{Problem P2,} \quad w = 2n(n + p + 2). \quad (69)$$

In conclusion, use of this supplementary sweep increases the CPU time; nevertheless, depending on the severity of the storage problem, this course of action might be desirable, and sometimes essential, with certain systems and certain computers.

4.7. Descent Properties. The functions  $A(t)$ ,  $B(t)$ ,  $C$  solving Eqs. (45)-(52) are such that the following first-variation properties hold:<sup>9</sup>

$$\text{gradient phase,} \quad \delta I = \delta J = -\alpha Q; \quad (70)$$

$$\text{restoration phase,} \quad \delta P = -2\alpha P. \quad (71)$$

In Eq. (70),  $Q$  denotes the error in the optimality conditions (16) at the beginning of the gradient phase. Because of

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<sup>9</sup>This can be seen by substitution of (45)-(52) into (25)-(27).



(37)-(40), the error  $Q$  reduces to

$$Q = \int_0^1 B^T B dt + C^T C . \quad (72)$$

In Eq. (71),  $P$  denotes the constraint error (15) at the beginning of the restoration phase.

Note that the first variation properties (70) and (71) are consistent with the requirements (24). This being the case, it is possible in principle to determine the gradient stepsize so that the descent property (23-1) or (23-2) is enforced in the gradient phase. It is also possible to determine the restoration stepsize so that the descent property (23-3) is enforced in the restoration phase. For details, see the following section.

## 5. Determination of the Stepsizes

5.1. Gradient Stepsize. Suppose that the perturbations  $A(t)$ ,  $B(t)$ ,  $C$  solving the LTP-BVP (45)-(52) for the gradient phase<sup>10</sup> are known. Since the nominal functions  $x(t)$ ,  $u(t)$ ,  $\pi$  are known, the one-parameter family of varied functions (22) can be formed. After substitution of Eqs. (22) into Eqs. (1), (9), (15), the following functions of the stepsize are obtained:

$$\tilde{I} = \tilde{I}(\alpha), \quad \tilde{J} = \tilde{J}(\alpha), \quad \tilde{P} = \tilde{P}(\alpha). \quad (73)$$

Then, a one-dimensional search scheme is applied to (73-2), and a value of the stepsize  $\alpha$  is selected for which the following relations are satisfied:

$$\tilde{J}(\alpha) \leq \tilde{J}(0), \quad \tilde{P}(\alpha) \leq P_*, \quad \tilde{\tau}(\alpha) \geq 0, \quad (74)$$

where  $\tau$  is the final time and  $P_*$  is a preselected number, not necessarily small. Satisfaction of Ineq. (74-1) is possible because of the descent property of the gradient phase. Ineq. (74-2) is introduced to prevent excessive constraint

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<sup>10</sup>Therefore, the constants  $k_g$  and  $k_r$  are given by Eqs. (53).

violation. And Ineq. (74-3) is required for problems with free final time.

Prior to the satisfaction of (74), a scanning process is employed, leading to the bracketing of the minimum point for  $\tilde{J}(\alpha)$ . This operation is then followed by a Hermitian cubic interpolation process (Ref. 34), which is stopped whenever the following relation is satisfied:<sup>11</sup>

$$|\tilde{J}_\alpha(\alpha)| \leq \epsilon_3 \quad \text{or} \quad |\tilde{J}_\alpha(\alpha)/\tilde{J}_\alpha(0)| \leq \epsilon_4, \quad (75)$$

subject to an upper limit for the number of search steps  $N_s$ . Once a stepsize  $\alpha_0$  has been selected consistently with either (75) or the prescribed upper limit for the number of search steps, Ineqs. (74) must be checked. If satisfaction occurs, then the stepsize  $\alpha_0$  is accepted. If any violation occurs, then the stepsize  $\alpha_0$  must be bisected progressively until satisfaction of (74) is finally achieved.

5.2. Remark. Alternatively, the search for the gradient stepsize can be performed by replacing the augmented functional  $J$  with the augmented penalty functional  $W$ , defined by

$$W = J + kP, \quad (76)$$

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<sup>11</sup>

The symbols  $\epsilon_3$  and  $\epsilon_4$  denote small, preselected numbers.

where  $k$  denotes the penalty constant, to be suitably chosen. After substitution of Eqs. (22) into (76), the following function of the stepsize is obtained:

$$\tilde{W} = \tilde{W}(\alpha). \quad (77)$$

Then, the combination of scanning process/Hermitian cubic interpolation process leading to satisfaction of (75) is replaced by a combination of scanning process/Hermitian cubic interpolation leading to satisfaction of the following relation:<sup>12</sup>

$$|\tilde{W}_\alpha(\alpha)| \leq \varepsilon_5 \quad \text{or} \quad |\tilde{W}_\alpha(\alpha)/\tilde{W}_\alpha(0)| \leq \varepsilon_6, \quad (78)$$

subject to an upper limit for the number of search steps  $N_s$ . Once a stepsize  $\alpha_0$  has been selected consistently with either (78) or the prescribed upper limit for the number of search steps, Ineqs. (74) must be checked. If satisfaction occurs, then the stepsize  $\alpha_0$  is accepted. If any violation occurs, then the stepsize  $\alpha_0$  must be bisected progressively until satisfaction of (74) is finally achieved.

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<sup>12</sup>The symbols  $\varepsilon_5$  and  $\varepsilon_6$  denote small, preselected numbers.



5.3. Restoration Stepsize. Suppose that the perturbations  $A(t)$ ,  $B(t)$ ,  $C$  solving the LTP-BVP (45)-(52) for the restoration phase<sup>13</sup> are known. Since the nominal functions  $x(t)$ ,  $u(t)$ ,  $\pi$  are known, the one-parameter family of varied functions (22) can be formed. For this one-parameter family, the constraint error (15) becomes a function of the form

$$\tilde{P} = \tilde{P}(\alpha). \quad (79)$$

Then, the stepsize  $\alpha$  must be selected so that the following relations are satisfied:

$$\tilde{P}(\alpha) < \tilde{P}(0), \quad \tilde{I}(\alpha) \geq 0. \quad (80)$$

Satisfaction of Ineq. (80-1) is possible because of the descent property of the restoration phase. Ineq. (80-2) is required for problems with free final time.

In order to achieve satisfaction of Ineqs. (80), a bisection process is applied to the restoration stepsize  $\alpha$ , starting from the reference stepsize  $\alpha_0 = 1$ . This reference stepsize has the property of yielding one-step restoration for the case where the constraints (2)-(5) are linear.

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<sup>13</sup>Therefore, the constants  $k_g$  and  $k_r$  are given by Eqs. (54).

#### 5.4. Iterative Procedure for the Restoration Phase.

The descent property (71) of the restoration phase guarantees satisfaction of Ineq. (80-1) at the end of any iteration, but not satisfaction of Ineq. (19-1). Therefore, the restoration algorithm must be employed iteratively until Ineq. (19-1) is satisfied. At this point, the restoration phase is terminated.

5.5. Descent Property of a Cycle. A descent property exists for a complete gradient-restoration cycle under the assumption of small stepsizes. Let  $\alpha_g$  denote the gradient stepsize and  $\alpha_r$  the restoration stepsize. Simple manipulations, omitted for the sake of brevity, show that the gradient corrections are of  $O(\alpha_g)$ , while the restoration corrections are of  $O(\alpha_r \alpha_g^2)$ . Hence, for  $\alpha_g$  sufficiently small, the restoration corrections are negligible with respect to the gradient corrections. Therefore, the restoration phase preserves the descent property of the gradient phase.

More specifically, let  $I$ ,  $\tilde{I}$ ,  $\hat{I}$  denote the values of the functional (1) at the beginning of the gradient phase, at the end of the gradient phase, and at the end of the subsequent restoration phase. Note that  $I$  and  $\tilde{I}$  are not comparable, since the constraints are not satisfied to the same accuracy. On the other hand,  $I$  and  $\hat{I}$  are comparable, and the gradient stepsize  $\alpha_g$  can be selected so that

$$\hat{I} < I. \quad (81)$$

This inequality constitutes the descent property of a complete gradient-restoration cycle. In order to enforce it, one proceeds as follows. At the end of the restoration phase, one must verify Ineq. (81). If it is satisfied, the next gradient phase is started; otherwise, the previous gradient stepsize is bisected as many times as needed until, after restoration, Ineq. (81) is satisfied.

## 6. Summary of the Algorithm

This algorithm includes cycles composed of a gradient phase and a restoration phase. The objective of each cycle is to decrease the functional  $I$  so that Ineq. (81) is satisfied, while the constraints are satisfied to a predetermined accuracy (19-1).

6.1. Gradient Phase. This phase involves a single iteration, and its objective is to decrease the augmented functional  $J$ , while the constraints are satisfied to first order. The gradient phase can be summarized as follows.

(a) Assume nominal functions  $x(t)$ ,  $u(t)$ ,  $\pi$  which satisfy the constraints (2)-(5) within the preselected accuracy (19-1).

(b) For the nominal functions, compute the vectors  $f_x$ ,  $f_u$ ,  $f_\pi$  and the matrices  $\phi_x$ ,  $\phi_u$ ,  $\phi_\pi$  and  $S_x$ ,  $S_u$ ,  $S_\pi$  along the interval of integration. At the final point, compute the vectors  $g_x$ ,  $g_\pi$  and the matrices  $\psi_x$ ,  $\psi_\pi$ .

(c) Solve the LTP-BVP (45)-(52), with constants  $k_g$  and  $k_r$  given by Eqs. (53), using the method of particular solutions. In this way, obtain the functions  $A(t)$ ,  $B(t)$ ,  $C$  and the multipliers  $\lambda(t)$ ,  $\rho(t)$ ,  $\mu$ .

(d) Using the functions in (c), compute the gradient stepsize by a one-dimensional search on the augmented functional  $\tilde{J}(\alpha)$  until satisfaction of Ineq. (75) occurs. Then, bisect the resulting stepsize  $\alpha_0$  (if necessary), until



satisfaction of Ineqs. (74) occurs.

(e) Once the gradient stepsize is known, compute the varied functions  $\tilde{x}(t)$ ,  $\tilde{u}(t)$ ,  $\tilde{\pi}$  with Eqs. (22).

6.2. Restoration Phase. This phase involves one or more iterations, and its objective is to reduce the constraint error  $P$ , until satisfaction of (19-1) occurs. Within a single iteration, the objective is to decrease the constraint error to a level compatible with Ineq. (23-3), while the norm squared of the variations of the control and the parameter is minimized.

The nominal functions  $x(t)$ ,  $u(t)$ ,  $\pi$  are chosen as follows: for the first restorative iteration, the nominal functions are identical with the varied functions obtained at the end of the previous gradient iteration; for any subsequent restorative iteration, the nominal functions are identical with the varied functions obtained at the end of the previous restorative iteration. With this understanding, the restoration phase can be summarized as follows.

(a) Assume nominal functions  $x(t)$ ,  $u(t)$ ,  $\pi$  which satisfy condition (3), but violate at least one of conditions (2) and (4)-(5).

(b) For the nominal functions, compute the vectors  $(\dot{x} - \phi)$ ,  $S$  and the matrices  $\phi_x$ ,  $\phi_u$ ,  $\phi_\pi$  and  $S_x$ ,  $S_u$ ,  $S_\pi$  along the interval of integration. At the final point, compute

the vector  $\psi$  and the matrices  $\psi_x, \psi_\pi$ .

(c) Solve the LTP-BVP (45)-(52), with constants  $k_g$  and  $k_r$  given by Eqs. (54), using the method of particular solutions. In this way, obtain the functions  $A(t)$ ,  $B(t)$ ,  $C$  and the multipliers  $\lambda(t)$ ,  $\rho(t)$ ,  $\mu$ .

(d) Using the functions in (c), compute the restoration stepsize by a one-dimensional search on the constraint error  $\tilde{P}(\alpha)$ . To this effect, perform a bisection process on  $\alpha$ , starting from  $\alpha_0 = 1$ , until Ineqs. (80) are satisfied.

(e) Once the restoration stepsize is known, compute the varied functions  $\tilde{x}(t)$ ,  $\tilde{u}(t)$ ,  $\tilde{\pi}$  with Eqs. (22).

(f) Verify whether the varied functions in (e) satisfy Ineq. (19-1). If this is the case, the restoration phase is terminated. Otherwise, return to (a) and continue the process until satisfaction of (19-1) occurs.

6.3. Gradient-Restoration Cycle. After the restoration phase is completed, verify whether Ineq. (81) is satisfied. If this is the case, start the next cycle of the sequential gradient-restoration algorithm. If not, return to the previous gradient phase and reduce the gradient stepsize (using a bisection process) until, after restoration, Ineq. (81) is satisfied.

6.4. Computational Considerations. Here, special conditions relevant to the computer implementation of the sequential gradient-restoration algorithm are presented.

Starting Condition. The present algorithm can be started with nominal functions  $x(t)$ ,  $u(t)$ ,  $\pi$  satisfying condition (3) and violating none, some, or all of conditions (2) and (4)-(5). If the nominal functions are such that Ineq. (19-1) is violated, the algorithm starts with a restoration phase; hence, the first cycle is a half cycle, involving a restoration phase only. On the other hand, if the nominal functions are such that Ineq. (19-1) is satisfied, the algorithm starts with a gradient phase; hence, the first cycle is a complete gradient-restoration cycle.

Bypassing Condition. At the end of the gradient phase of any cycle, the constraint error  $P$  must be computed. If Ineq. (19-1) is violated, a restoration phase is started. Otherwise, the restoration phase is bypassed, and the next gradient phase of the algorithm is started.

Stopping Conditions. For the restoration phase taken individually, convergence is achieved whenever Ineq. (19-1) is satisfied. For the sequential gradient-restoration algorithm taken as a whole, convergence is achieved whenever Ineqs. (19-1) and (19-2) are satisfied simultaneously.

## 7. Experimental Conditions

In order to evaluate the theory, several examples were solved. The sequential gradient-restoration algorithms associated with Problems P1 and P2 were programmed in FORTRAN IV, and the numerical results were obtained in double-precision arithmetic.

Computations were performed at Rice University using an IBM 370/155 computer. For each example, the interval of integration was divided into 100 steps. The differential equations were integrated using Hamming's modified predictor-corrector method with a special Runge-Kutta starting procedure (Ref. 35). The definite integrals I, J, P, Q were computed using a modified Simpson's rule. The method of particular solutions (Ref. 7) was used to solve the linear, two-point boundary-value problems associated with both the gradient phase and the restoration phase.

7.1. Convergence Conditions. The parameters  $\epsilon_1$ ,  $\epsilon_2$ ,  $\epsilon_4$  appearing in Ineqs. (19) and (75) were set at the levels<sup>14</sup>

$$\epsilon_1 = E-08, \quad \epsilon_2 = E-04, \quad \epsilon_4 = E-03. \quad (82)$$

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<sup>14</sup>The symbol E<sub>±</sub>ab stands for 10<sup>±ab</sup>.



The tolerance level (82-1) characterizes the restoration phase; the tolerance levels (82-1) and (82-2), employed in combination, characterize the algorithm as a whole; and the tolerance level (82-3) characterizes the one-dimensional search for the gradient stepsize.

7.2. Safeguards. For the gradient phase, the parameter  $P_*$  appearing in Ineq. (74-2) was set at the level

$$P_* = 10. \quad (83)$$

The tolerance level (83) limits the constraint violation which is permissible during the gradient phase. Also for the gradient phase, the number of Hermitian search steps required to satisfy Ineq. (75) was subject to the upper bound

$$N_s \leq 5. \quad (84)$$

7.3. Nonconvergence Conditions. The sequential gradient-restoration algorithms were programmed to stop whenever violation of any of the following inequalities occurred:<sup>15</sup>

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<sup>15</sup>Inequality (87) is characteristic of the IBM 370/155 computer.

$$N_c \leq 30, \quad N \leq 100, \quad N_r \leq 10, \quad (85)$$

$$N_{bg} \leq 10, \quad N_{br} \leq 10, \quad N_{bc} \leq 5, \quad (86)$$

$$M \leq 0.83 E+75. \quad (87)$$

Here,  $N_c$  is the number of cycles,  $N$  is the total number of iterations,  $N_r$  is the number of restorative iterations per cycle,  $N_{bg}$  is the number of bisections of the gradient stepsize required to satisfy Ineqs. (74),  $N_{br}$  is the number of bisections of the restoration stepsize required to satisfy Ineqs. (80),  $N_{bc}$  is the number of bisections of the gradient stepsize required to satisfy Ineq. (81), and  $M$  is the modulus of any of the quantities employed in the algorithm.

### 8. Numerical Examples, Problem P1

In this section, four numerical examples are described employing scalar notation. In particular, the symbols  $x_i(t)$ ,  $i=1, \dots, n$ , denote the components of the state; the symbols  $u_i(t)$ ,  $i=1, \dots, m$ , denote the components of the control; and the symbols  $\pi_i$ ,  $i=1, \dots, p$ , denote the components of the parameter.

For all of the examples, a time normalization is used in order to simplify the numerical computations. Specifically, the actual time  $\theta$  is replaced by the normalized time

$$t = \theta/\tau, \quad (88)$$

which is defined in such a way that  $t=0$  at the initial point and  $t=1$  at the final point. The actual final time  $\tau$ , if it is free, is regarded as a component of the vector parameter  $\pi$  to be optimized. In this way, an optimal control problem with variable final time is converted into an optimal control problem with fixed final time.

Concerning the convergence history, the terminology is as follows:  $N_c$  denotes the cycle number,  $N_g$  is the number of gradient iterations per cycle,  $N_r$  is the number of restorative iterations per cycle,  $N$  is the total number of iterations,  $P$  is the constraint error,  $Q$  is the error in the optimality conditions, and  $I$  the value of the functional being minimized.

Example 8.1. This example involves (i) a quadratic functional, (ii) nonlinear differential equations, (iii) boundary conditions of the fixed endpoint type, and (iv) fixed final time  $\tau = 1$ :

$$I = \int_0^1 (1 + x_1^2 + x_2^2 + u_1^2) dt, \quad (89)$$

$$\dot{x}_1 = u_1 - x_2^2, \quad \dot{x}_2 = u_1 - x_1 x_2, \quad (90)$$

$$x_1(0) = 0, \quad x_2(0) = 1, \quad (91)$$

$$x_1(1) = 1, \quad x_2(1) = 2. \quad (92)$$

The assumed nominal functions are

$$x_1(t) = t, \quad x_2(t) = 1 + t, \quad u_1(t) = 1. \quad (93)$$

The numerical results are given in Tables 1-2. Convergence to the desired stopping condition occurs in  $N_c = 3$  cycles and  $N = 7$  iterations, which include 2 gradient iterations and 5 restorative iterations.

Example 8.2. This example involves (i) a nonquadratic functional, (ii) nonlinear differential equations, (iii) boundary conditions of the fixed endpoint type, and (iv) fixed final time  $\tau = 1$ :



$$I = \int_0^1 (-2\cos u_1) dt, \quad (94)$$

$$\dot{x}_1 = 2\sin u_1 - 1, \quad \dot{x}_2 = x_1, \quad (95)$$

$$x_1(0) = 0, \quad x_2(0) = 0, \quad (96)$$

$$x_1(1) = 0, \quad x_2(1) = 0.3. \quad (97)$$

The assumed nominal functions are

$$x_1(t) = 0, \quad x_2(t) = 0.3t, \quad u_1(t) = 0. \quad (98)$$

The numerical results are given in Tables 3-4. Convergence to the desired stopping condition occurs in  $N_c = 6$  cycles and  $N = 13$  iterations, which include 5 gradient iterations and 8 restorative iterations.

Example 8.3. This example is a minimum time problem and involves (i) a linear functional, (ii) nonlinear differential equations, (iii) boundary conditions of the fixed final state type, and (iv) free final time  $\tau$ . After setting  $\pi_1 = \tau$ , the problem is as follows:

$$I = \pi_1, \quad (99)$$

$$\dot{x}_1 = \pi_1 u_1, \quad \dot{x}_2 = \pi_1 (u_1^2 - x_1^2), \quad (100)$$

$$x_1(0) = 0, \quad x_2(0) = 0, \quad (101)$$

$$x_1(1) = 1, \quad x_2(1) = 0. \quad (102)$$

The assumed nominal functions are

$$x_1(t) = t, \quad x_2(t) = 0, \quad u_1(t) = 1, \quad \pi_1 = 1. \quad (103)$$

The numerical results are given in Tables 5-6. Convergence to the desired stopping condition occurs in  $N_c = 3$  cycles and  $N = 7$  iterations, which include 2 gradient iterations and 5 restorative iterations.

Example 8.4. This example is a minimum time problem and involves (i) a linear functional, (ii) nonlinear differential equations, (iii) components of the final state partly given and partly free, and (iv) free final time  $\tau$ . After setting  $\pi_1 = \tau$ , the problem is as follows:

$$I = \pi_1, \quad (104)$$

$$\dot{x}_1 = \pi_1 x_3 \cos u_1, \quad \dot{x}_2 = \pi_1 x_3 \sin u_1, \quad \dot{x}_3 = \pi_1 \sin u_1, \quad (105)$$

$$x_1(0) = 0, \quad x_2(0) = 0, \quad x_3(0) = 0, \quad (106)$$

$$x_1(1) = 1. \quad (107)$$

The assumed nominal functions are

$$x_1(t)=t, \quad x_2(t)=0, \quad x_3(t)=0, \quad u_1(t)=1, \quad \pi_1=1. \quad (108)$$

The numerical results are given in Tables 7-8. Convergence to the desired stopping condition occurs in  $N_c = 4$  cycles and  $N = 12$  iterations, which include 3 gradient iterations and 9 restorative iterations.

### 9. Numerical Examples, Problem P2

In this section, four numerical examples are described employing scalar notation. The symbols used for Problem P2 are the same as those employed for Problem P1. The time normalization (88) is employed.

Example 9.1. This example involves (i) a quadratic functional, (ii) a nonlinear differential equation, (iii) a state inequality constraint of the first order<sup>16</sup>, (iv) boundary conditions of the fixed endpoint type, and (v) fixed final time  $\tau = 1$ :

$$I = \int_0^1 (x_1^2 + u_1^2) dt, \quad (109)$$

$$\dot{x}_1 = x_1^2 - u_1, \quad (110)$$

$$x_1 - 0.9 \geq 0, \quad (111)$$

$$x_1(0) = 1, \quad (112)$$

$$x_1(1) = 1. \quad (113)$$

---

<sup>16</sup>This means that the first time derivative of the left-hand side of Ineq. (111) contains the control explicitly.



Upon introducing the auxiliary state variable  $x_2$  and the auxiliary control variable  $u_2$  defined by (Ref. 25)

$$x_1 - 0.9 = x_2^2, \quad \dot{x}_2 = u_2, \quad (114)$$

we replace the inequality constrained problem (109)-(113) with the following equality constrained problem:

$$I = \int_0^1 (x_1^2 + u_1^2) dt, \quad (115)$$

$$\dot{x}_1 = x_1^2 - u_1, \quad \dot{x}_2 = u_2, \quad (116)$$

$$x_1^2 - u_1 - 2x_2u_2 = 0, \quad (117)$$

$$x_1(0) = 1, \quad x_2(0) = \sqrt{0.1}, \quad (118)$$

$$x_1(1) = 1. \quad (119)$$

The assumed nominal functions are

$$x_1(t) = 1, \quad x_2(t) = \sqrt{0.1}, \quad u_1(t) = 1, \quad u_2(t) = 1. \quad (120)$$

The numerical results are given in Tables 9-10. Convergence to the desired stopping condition occurs in  $N_c = 5$  cycles and  $N = 12$  iterations, which include 4 gradient iterations and 8 restorative iterations.

Example 9.2. This example involves (i) a quadratic functional, (ii) linear differential equations, (iii) a state inequality constraint of the second order<sup>17</sup>, (iv) boundary conditions of the fixed endpoint type, and (v) fixed final time  $\tau = 1$ :

$$I = \int_0^1 u_1^2 dt, \quad (121)$$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u_1, \quad (122)$$

$$0.15 - x_1 \geq 0, \quad (123)$$

$$x_1(0) = 0, \quad x_2(0) = 1, \quad (124)$$

$$x_1(1) = 0, \quad x_2(1) = -1. \quad (125)$$

Upon introducing the auxiliary state variables  $x_3$ ,  $x_4$  and the auxiliary control variable  $u_2$  defined by (Ref. 25)

$$0.15 - x_1 = x_3^2, \quad \dot{x}_3 = x_4, \quad \dot{x}_4 = u_2, \quad (126)$$

---

<sup>17</sup>This means that the second time derivative of the left-hand side of Ineq. (123) contains the control explicitly, while this is not the case with the first time derivative.

we replace the inequality constrained problem (121)-(125) with the following equality constrained problem:

$$I = \int_0^1 u_1^2 dt, \quad (127)$$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u_1, \quad \dot{x}_3 = x_4, \quad \dot{x}_4 = u_2, \quad (128)$$

$$u_1 + 2x_3u_2 + 2x_4^2 = 0, \quad (129)$$

$$x_1(0) = 0, \quad x_2(0) = 1, \quad x_3(0) = \sqrt{0.15}, \quad x_4(0) = -1/\sqrt{0.60}, \quad (130)$$

$$x_1(1) = 0, \quad x_2(1) = -1. \quad (131)$$

The assumed nominal functions are

$$x_1(t) = 0, \quad x_2(t) = 1 - 2t, \quad x_3(t) = \sqrt{0.15}(1 - 2t), \quad (132)$$

$$x_4(t) = (2t - 1)/\sqrt{0.60}, \quad u_1(t) = 1, \quad u_2(t) = 0. \quad (133)$$

The numerical results are given in Tables 11-12. Convergence to the desired stopping condition occurs in  $N_c = 8$  cycles and  $N = 16$  iterations, which include 7 gradient iterations and 9 restorative iterations.

Example 9.3. This example is a minimum time problem and involves (i) a linear functional, (ii) nonlinear differential equations, (iii) a state-derivative inequality constraint,

(iv) boundary conditions of the fixed final state type, and (v) free final time  $\tau$ . After setting  $\pi_1 = \tau$ , the problem is as follows:

$$I = \pi_1, \quad (134)$$

$$\dot{x}_1 = \pi_1 u_1, \quad \dot{x}_2 = \pi_1 (u_1^2 - x_1^2 - 0.5), \quad (135)$$

$$\dot{x}_2 / \pi_1 + 0.5 \geq 0, \quad (136)$$

$$x_1(0) = 0, \quad x_2(0) = 0, \quad (137)$$

$$x_1(1) = 1, \quad x_2(1) = -\pi/4. \quad (138)$$

Upon introducing the auxiliary control variable  $u_2$  defined by (Ref. 25)

$$\dot{x}_2 / \pi_1 + 0.5 - u_2^2 = 0, \quad (139)$$

we replace the inequality constrained problem (134)-(138) with the following equality constrained problem:

$$I = \pi_1, \quad (140)$$

$$\dot{x}_1 = \pi_1 u_1, \quad \dot{x}_2 = \pi_1 (u_1^2 - x_1^2 - 0.5), \quad (141)$$

$$u_1^2 - x_1^2 - u_2^2 = 0, \quad (142)$$

$$x_1(0) = 0, \quad x_2(0) = 0, \quad (143)$$



$$x_1(1) = 1, \quad x_2(1) = -\pi/4. \quad (144)$$

The assumed nominal functions are

$$x_1(t) = t, \quad x_2(t) = -(\pi/4)t, \quad u_1(t) = 1, \quad u_2(t) = 1, \quad \pi_1 = 1. \quad (145)$$

The numerical results are given in Tables 13-14. Convergence to the desired stopping condition occurs in  $N_c = 6$  cycles and  $N = 14$  iterations, which include 5 gradient iterations and 9 restorative iterations.

Example 9.4. This example involves (i) a quadratic functional, (ii) linear differential equations, (iii) a control inequality constraint, (iv) boundary conditions of the fixed endpoint type, and (v) fixed final time  $\tau$ :

$$I = \int_0^1 (1 + x_1^2 + x_2^2 + u_1^2) dt, \quad (146)$$

$$\dot{x}_1 = u_1 - x_2, \quad \dot{x}_2 = u_1 - 2x_1, \quad (147)$$

$$6 - u_1 \geq 0, \quad (148)$$

$$x_1(0) = 0, \quad x_2(0) = 1, \quad (149)$$

$$x_1(1) = 1, \quad x_2(1) = 2. \quad (150)$$

Upon introducing the auxiliary control variable  $u_2$  defined by

$$6 - u_1 - u_2^2 = 0, \quad (151)$$

we replace the inequality constrained problem (146)-(150) with the following equality constrained problem:

$$I = \int_0^1 (1 + x_1^2 + x_2^2 + u_1^2) dt, \quad (152)$$

$$\dot{x}_1 = u_1 - x_2, \quad \dot{x}_2 = u_1 - 2x_1, \quad (153)$$

$$6 - u_1 - u_2^2 = 0, \quad (154)$$

$$x_1(0) = 0, \quad x_2(0) = 1, \quad (155)$$

$$x_1(1) = 1, \quad x_2(1) = 2. \quad (156)$$

The assumed nominal functions are

$$x_1(t) = 5t - 4t^2, \quad x_2(t) = 1 + 5t - 4t^2, \quad (157)$$

$$u_1(t) = 6(1 - t), \quad u_2(t) = 2t. \quad (158)$$

The numerical results are given in Tables 15-16. Convergence to the desired stopping condition occurs in  $N_c = 11$  cycles and  $N = 24$  iterations, which include 10 gradient iterations and 14 restorative iterations.

Table 1. Convergence history, Example 8.1.

$N_c$	$N_g$	$N_r$	$N$	$P$	$Q$	$I$
0	0	0	0	0.72E+01		
1	0	4	4	0.32E-10	0.97E+00	33.67701
2	1	1	6	0.84E-13	0.50E-02	33.46606
3	1	0	7	0.51E-09	0.41E-04	33.46484

Table 2. Converged solution, Example 8.1.

$t$	$x_1$	$x_2$	$u_1$
0.0	0.0000	1.0000	-8.3428
0.1	-0.7862	0.2778	-6.3676
0.2	-1.3011	-0.2366	-3.8632
0.3	-1.5837	-0.5625	-1.4845
0.4	-1.6735	-0.7169	0.4682
0.5	-1.6003	-0.7107	1.9931
0.6	-1.3780	-0.5437	3.2522
0.7	-1.0080	-0.2055	4.4920
0.8	-0.4877	0.3179	6.0526
0.9	0.1807	1.0416	8.4996
1.0	1.0000	2.0000	13.0496

$\tau = 1.00000$

Table 3. Convergence history, Example 8.2.

$N_c$	$N_g$	$N_r$	$N$	$P$	$Q$	$I$
0	0	0	0	0.10E+01		
1	0	4	4	0.17E-08	0.67E+00	-1.11665
2	1	2	7	0.17E-11	0.34E-01	-1.16519
3	1	1	9	0.11E-10	0.30E-02	-1.16923
4	1	1	11	0.58E-15	0.64E-03	-1.16950
5	1	0	12	0.20E-08	0.18E-03	-1.16961
6	1	0	13	0.36E-08	0.50E-04	-1.16964

Table 4. Converged solution, Example 8.2.

$t$	$x_1$	$x_2$	$u_1$
0.0	0.0000	0.0000	1.3333
0.1	0.0937	0.0047	1.3049
0.2	0.1856	0.0186	1.2609
0.3	0.2742	0.0417	1.2005
0.4	0.3575	0.0733	1.1131
0.5	0.4309	0.1128	0.9784
0.6	0.4842	0.1589	0.7517
0.7	0.4921	0.2082	0.3661
0.8	0.4141	0.2544	-0.1521
0.9	0.2381	0.2877	-0.6087
1.0	0.0000	0.3000	-0.8959
$\tau = 1.00000$			



Table 5. Convergence history, Example 8.3.

$N_c$	$N_g$	$N_r$	$N$	$P$	$Q$	$I$
0	0	0	0	0.53E+00		
1	0	4	4	0.74E-16	0.53E-01	1.58101
2	1	1	6	0.33E-08	0.13E-03	1.57080
3	1	0	7	0.28E-08	0.16E-05	1.57075

Table 6. Converged solution, Example 8.3.

$t$	$x_1$	$x_2$	$u_1$
0.0	0.0000	0.0000	0.9997
0.1	0.1564	0.1544	0.9874
0.2	0.3089	0.2937	0.9508
0.3	0.4538	0.4043	0.8907
0.4	0.5876	0.4752	0.8087
0.5	0.7069	0.4997	0.7067
0.6	0.8087	0.4752	0.5875
0.7	0.8907	0.4042	0.4538
0.8	0.9507	0.2937	0.3092
0.9	0.9874	0.1544	0.1572
1.0	1.0000	0.0000	0.0017

$$\tau = \pi_1 = 1.57075$$

Table 7. Convergence history, Example 8.4.

$N_c$	$N_g$	$N_r$	$N$	$P$	$Q$	$I$
0	0	0	0	0.17E+01		
1	0	5	5	0.44E-16	0.25E+00	1.83370
2	1	2	8	0.29E-09	0.42E-01	1.78266
3	1	1	10	0.39E-09	0.62E-03	1.77262
4	1	1	12	0.10E-16	0.92E-05	1.77245

Table 8. Converged solution, Example 8.4.

$t$	$x_1$	$x_2$	$x_3$	$u_1$
0.0	0.0000	0.0000	0.0000	1.5708
0.1	0.0016	0.0155	0.1765	1.4133
0.2	0.0129	0.0607	0.3486	1.2558
0.3	0.0425	0.1311	0.5121	1.0984
0.4	0.0974	0.2198	0.6630	0.9412
0.5	0.1819	0.3180	0.7975	0.7841
0.6	0.2976	0.4161	0.9123	0.6270
0.7	0.4428	0.5046	1.0046	0.4699
0.8	0.6132	0.5748	1.0722	0.3126
0.9	0.8018	0.6196	1.1132	0.1549
1.0	1.0000	0.6346	1.1266	-0.0033

$$\tau = \pi_1 = 1.77245$$

Table 9. Convergence history, Example 9.1.

$N_c$	$N_g$	$N_r$	$N$	$P$	$Q$	$I$
0	0	0	0	0.14E+01		
1	0	3	3	0.52E-09	0.35E+00	1.83569
2	1	2	6	0.15E-16	0.14E-01	1.66599
3	1	1	8	0.10E-09	0.24E-03	1.65742
4	1	1	10	0.60E-17	0.15E-03	1.65697
5	1	1	12	0.96E-18	0.98E-04	1.65678

Table 10. Converged solution, Example 9.1.

$t$	$x_1$	$x_2$	$u_1$	$u_2$
0.0	1.0000	0.3162	1.7482	-1.1831
0.1	0.9410	0.2025	1.3353	-1.1104
0.2	0.9095	0.0978	1.0097	-0.9324
0.3	0.9006	0.0246	0.8366	-0.5177
0.4	0.9000	-0.0090	0.8067	-0.1865
0.5	0.9003	-0.0177	0.8104	-0.0018
0.6	0.9000	-0.0094	0.8135	0.1816
0.7	0.9005	0.0238	0.7864	0.5158
0.8	0.9094	0.0972	0.6442	0.9398
0.9	0.9409	0.2024	0.4360	1.1097
1.0	1.0000	0.3162	0.2470	1.1904

$\tau = 1.00000$

Table 11. Convergence history, Example 9.2.

$N_c$	$N_g$	$N_r$	$N$	$P$	$Q$	$I$
0	0	0	0	0.22E+02		
1	0	5	5	0.44E-13	0.11E+00	6.03009
2	1	1	7	0.15E-14	0.79E-02	5.93793
3	1	1	9	0.28E-17	0.20E-02	5.93016
4	1	1	11	0.12E-18	0.74E-03	5.92817
5	1	1	13	0.15E-20	0.37E-03	5.92738
6	1	0	14	0.86E-08	0.20E-03	5.92687
7	1	0	15	0.62E-08	0.12E-03	5.92661
8	1	0	16	0.74E-08	0.52E-04	5.92650

Table 12. Converged solution, Example 9.2.

$t$	$x_1$	$x_2$	$x_3$	$x_4$	$u_1$	$u_2$
0.0	0.0000	1.0000	0.3872	-1.2909	-4.4535	1.4461
0.1	0.0793	0.6045	0.2657	-1.1375	-3.4592	1.6392
0.2	0.1242	0.3078	0.1606	-0.9583	-2.4746	1.9862
0.3	0.1442	0.1105	0.0756	-0.7301	-1.4669	2.6475
0.4	0.1496	0.0145	0.0175	-0.4152	-0.4730	3.6484
0.5	0.1499	-0.0001	-0.0045	-0.0174	0.0375	4.1943
0.6	0.1497	-0.0118	0.0147	0.4010	-0.4405	4.0278
0.7	0.1446	-0.1097	0.0733	0.7483	-1.5262	2.7693
0.8	0.1244	-0.3098	0.1599	0.9688	-2.4681	1.8476
0.9	0.0794	-0.6057	0.2655	1.1403	-3.4508	1.6007
1.0	0.0000	-1.0000	0.3872	1.2909	-4.4354	1.4228

$\tau = 1.00000$



Table 13. Convergence history, Example 9.3.

$N_c$	$N_g$	$N_r$	$N$	$P$	$Q$	$I$
0	0	0	0	0.11E+01		
1	0	5	5	0.22E-14	0.21E-01	1.82848
2	1	2	8	0.47E-15	0.20E-02	1.82290
3	1	1	10	0.83E-12	0.55E-03	1.82245
4	1	1	12	0.18E-13	0.22E-03	1.82234
5	1	0	13	0.60E-08	0.10E-03	1.82224
6	1	0	14	0.77E-08	0.39E-04	1.82222

Table 14. Converged solution, Example 9.3.

$t$	$x_1$	$x_2$	$u_1$	$u_2$
0.0	0.0000	0.0000	0.4999	0.4999
0.1	0.0905	-0.0465	0.4916	0.4832
0.2	0.1781	-0.0989	0.4670	0.4317
0.3	0.2598	-0.1623	0.4271	0.3389
0.4	0.3331	-0.2401	0.3768	0.1762
0.5	0.4020	-0.3298	0.4020	0.0092
0.6	0.4824	-0.4209	0.4824	0.0000
0.7	0.5788	-0.5120	0.5788	0.0001
0.8	0.6945	-0.6031	0.6945	0.0000
0.9	0.8334	-0.6942	0.8333	-0.0007
1.0	1.0000	-0.7853	0.9996	0.0008

$\tau = \pi_1 = 1.82222$

Table 15. Convergence history, Example 9.4.

$N_c$	$N_g$	$N_r$	N	P	Q	I
0	0	0	0	0.38E+01		
1	0	4	4	0.18E-12	0.36E+00	20.27422
2	1	2	7	0.36E-10	0.37E-01	20.19329
3	1	1	9	0.79E-10	0.96E-02	20.18932
4	1	1	11	0.12E-11	0.49E-02	20.18813
5	1	1	13	0.32E-13	0.20E-02	20.18760
6	1	1	15	0.34E-14	0.12E-02	20.18733
7	1	1	17	0.20E-15	0.63E-03	20.18718
8	1	1	19	0.96E-16	0.50E-03	20.18707
9	1	1	21	0.45E-17	0.24E-03	20.18700
10	1	1	23	0.16E-16	0.34E-03	20.18693
11	1	0	24	0.29E-08	0.71E-04	20.18688

Table 16. Converged solution, Example 9.4.

t	$x_1$	$x_2$	$u_1$	$u_2$
0.0	0.0000	1.0000	6.0000	0.0000
0.1	0.4716	1.5519	6.0000	0.0000
0.2	0.8742	1.9967	5.3504	0.8059
0.3	1.1410	2.2746	4.3155	1.2978
0.4	1.2930	2.4172	3.4641	1.5924
0.5	1.3586	2.4610	2.7629	1.7991
0.6	1.3598	2.4346	2.1838	1.9535
0.7	1.3133	2.3603	1.7031	2.0728
0.8	1.2320	2.2549	1.3012	2.1676
0.9	1.1253	2.1315	0.9618	2.2445
1.0	1.0000	2.0000	0.6710	2.3084
$\tau = 1.00000$				

## 10. Discussion and Conclusions

In this paper, two members of the family of sequential gradient-restoration algorithms for the solution of optimal control problems are presented. These algorithms are of the ordinary-gradient type. One is associated with the solution of Problem P1, Eqs. (1)-(4), and the other is associated with the solution of Problem P2, Eqs. (1)-(5).

Problem P1 consists of minimizing a functional  $I$  which depends on the  $n$ -vector state  $x(t)$ , the  $m$ -vector control  $u(t)$ , and the  $p$ -vector parameter  $\pi$ . The state is given at the initial point. At the final point, the state and the parameter are required to satisfy  $q$  scalar relations. Along the interval of integration, the state, the control, and the parameter are required to satisfy  $n$  scalar differential equations. Problem P2 differs from Problem P1 in that the state, the control, and the parameter are required to satisfy  $k$  additional scalar relations along the interval of integration.

The importance of Problems P1 and P2 lies in the fact that a large number of problems of optimal control are covered by these formulations (Refs. 7-34). In particular, Problem P2 enlarges dramatically the number and variety of problems of optimal control which can be treated by gradient-restoration algorithms. Indeed, by suitable transformations, almost every known problem of optimal control can be brought into the

scheme of Problem P2. This statement applies, for instance, to the following situations: (i) problems with control equality constraints, (ii) problems with state equality constraints, (iii) problems with state-derivative equality constraints, (iv) problems with control inequality constraints, (v) problems with state inequality constraints, and (vi) problems with state-derivative inequality constraints. For an illustration of the scope and range of applicability of Problem P2, the reader is referred to Ref. 19 and Refs. 25-29.

The algorithms presented here include a sequence of two-phase cycles, composed of a gradient phase and a restoration phase. The gradient phase involves one iteration and is designed to decrease the value of the functional  $I$ , while the constraints are satisfied to first order. The restoration phase involves one or more iterations and is designed to force constraint satisfaction to a predetermined accuracy, while the norm squared of the variations of the control and the parameter is minimized, subject to the linearized constraints.

The principal property of the algorithms is that they produce a sequence of suboptimal solutions, each satisfying the constraints to the same predetermined accuracy. Therefore, the values of the functional  $I$  corresponding to any two elements of the sequence are comparable.

The gradient phase is characterized by a descent property



on the augmented functional  $J$ , which implies a descent property on the functional  $I$ . The restoration phase is characterized by a descent property on the constraint error  $P$ . The gradient stepsize and the restoration stepsize are chosen such that the restoration phase preserves the descent property of the gradient phase. Hence, the value of the functional  $I$  at the end of any complete gradient-restoration cycle is smaller than the value of the same functional at the beginning of that cycle.

Eight numerical examples are presented to illustrate the performance of the algorithms associated with Problem P1 and Problem P2. The numerical results show the feasibility as well as the convergence characteristics of these algorithms.

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